MATH 8

MIDTERM 1 SOLUTIONS

(1) Give, if possible, an example of a **true** conditional statement for which
 (a) the converse is true.
 (b) the converse is false.
 (c) the contrapositive is false.
 (d) the contrapositive is true.

The solutions here will vary. The key is to remember that if $P \Rightarrow Q$ is a conditional statement, then its **converse** is the conditional statement $Q \Rightarrow P$, and its **contraposition** is the conditional statement $\bar{Q} \Rightarrow \bar{P}$.

The following truth table should also help.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

(2) Let n be a natural number.
 (a) Prove that $n^2 + n + 3$ is odd.

Proof. First note that $n^2+n+3 = n(n+1)+3$. Next note that exactly one of n and $n+1$ is even and the other is odd. Since the product of an even integer and any other integer is even $n(n+1)$ must be even. Hence $n(n+1)+3$ is odd. □

 (b) Prove by contradiction that $\frac{n}{n+1} > \frac{n}{n+2}$.

Proof. Suppose, on the contrary, that $\frac{n}{n+1} \leq \frac{n}{n+2}$. By rearranging this inequality, this implies that $n(n+2) \leq n(n+1)$. Since $n \neq 0$, we can cancel n on both sides to get $n+2 \leq n+1$. Then, by subtracting n from both sides of the inequality, we obtain $2 \leq 1$. This is a contradiction. Hence, our assumption that $\frac{n}{n+1} \leq \frac{n}{n+2}$ must have been false. Therefore, $\frac{n}{n+1} > \frac{n}{n+2}$ is true. □

(3) Prove or give a counterexample.
 (a) $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ such that $x + y = 0$.
 This statement is true. We prove it by finding such a y for every $x \in \mathbb{R}$.

Proof. Let $x \in \mathbb{R}$, and let $y = -x$. Then $y \in \mathbb{R}$ and $x + y = x + (-x) = 0$. □

 (b) For all positive real numbers x, $2^x > x + 1$.
 This statement is false. The real number $x = 0$ is a counterexample.

(4) Prove that there do not exist integers m and n such that $12m + 15n = 1$.

Proof. Suppose, on the contrary, that there do exist integers m and n such that $12m+15n = 1$. This implies that $3(4m+5n) = 1$, and hence $4m+5n = \frac{1}{3}$. This is a contradiction because the left hand side of this equation is clearly an integer (since m and n are integers) and the right hand side is not an integer. Hence, our assumption that there do exist integers m and n such that $12m + 15n = 1$ must have been false, and therefore there do NOT exist such integers m and n. □